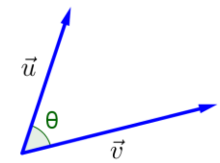
The Dot Product

(covers parts of Sullivan 9.5)

The **dot product** of $\vec{u} = \langle a_1, b_1 \rangle$ and $\vec{v} = \langle a_2, b_2 \rangle$ is: $\boldsymbol{\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2}$

Ex 1.
Suppose $\vec{u} = \langle 4, -2 \rangle$ and $\vec{v} = \langle -1, 3 \rangle$. Find $\vec{u} \cdot \vec{v}$.

The Dot Product Theorem
$\boldsymbol{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta}$ (here, θ is the angle between $\vec{u}$ and $\vec{v}$)



The Dot Product Theorem can help us find the angle between two vectors.

Ex 2.
Find the angle (in degrees) between $\vec{u} = \langle 4, -2 \rangle$ and $\vec{v} = \langle 1, 3 \rangle$.

Perpendicular vectors are said to be ___________________.

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if ______________.
Why? If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\theta = 90°$ so $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos 90° = 0$.
And if $\vec{u} \cdot \vec{v} = 0$, then $\|\vec{u}\| \|\vec{v}\| \cos \theta = 0$, so θ must be $90°$ and so $\vec{u}$ and $\vec{v}$ are orthogonal.

Ex 3.
Are $\langle 3, 5 \rangle$ and $\langle 2, -8 \rangle$ orthogonal?

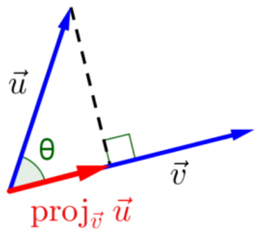
Ex 4.
Are $\langle 2, 1 \rangle$ and $\langle -1, 2 \rangle$ orthogonal?

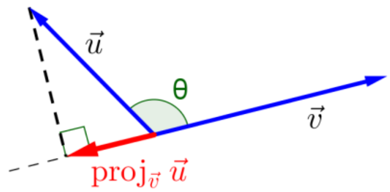
Ex 5.
A 3000-lb car is parked on a driveway that is inclined 15° to the horizontal. Find the magnitude of the force required to keep the car from rolling down the driveway.

Find the magnitude of the force experienced by the driveway due to the weight of the car. In other words, what is the magnitude of the force perpendicular to the driveway?

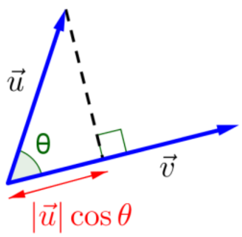
Projections and Components

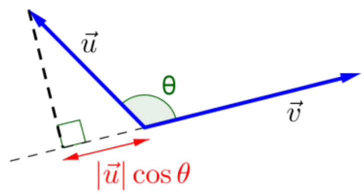
The **projection of $\vec{u}$ onto $\vec{v}$** (abbreviated $\mathbf{proj}_{\vec{v}}\,\vec{u}$) is basically the "shadow" of $\vec{u}$ cast upon $\vec{v}$.





How can we get the length of this vector? By looking at the right triangle. This signed length is also called the **component of $\vec{u}$ along $\vec{v}$**. It is calculated by $\|\vec{u}\|\cos\theta$.





Notice that we can also calculate the component of $\vec{u}$ along $\vec{v}$ by using the dot product: $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$

Why? Because $\|\vec{u}\|\cos\theta = \frac{\|\vec{u}\|\|\vec{v}\|\cos\theta}{\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$

With the component of $\vec{u}$ along $\vec{v}$, we can now calculate the projection of $\vec{u}$ onto $\vec{v}$:

projection of $\vec{u}$ onto $\vec{v}$ $= (\textbf{component of } \vec{u} \textbf{ along } \vec{v})(\textbf{unit vector in direction of } \vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\left(\frac{\vec{v}}{\|\vec{v}\|}\right)$

Ex 6.

Suppose $\vec{u} = \langle 2,1\rangle$ and $\vec{v} = \langle 3,-1\rangle$.

a) Find the component of $\vec{u}$ along $\vec{v}$.

b) Find the projection of $\vec{u}$ onto $\vec{v}$.

c) Decompose $\vec{u}$ into $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$, where $\overrightarrow{u_1}$ is parallel to $\vec{v}$ and $\overrightarrow{u_2}$ is orthogonal to $\vec{v}$.