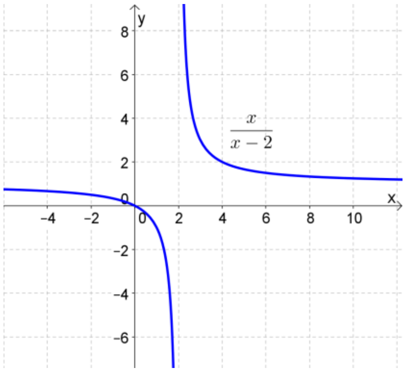
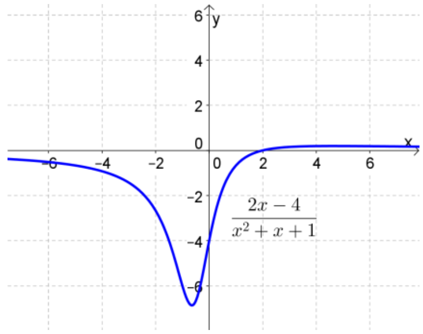
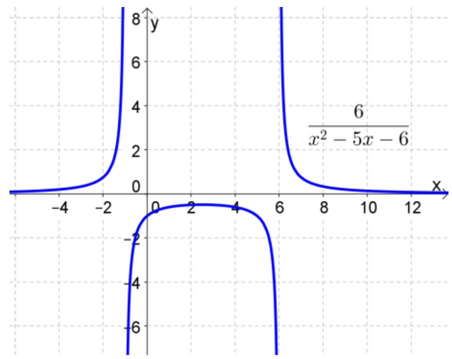
# Rational Functions

(covers parts of Sullivan 4.2 and 4.3)

A **rational function** is a polynomial over a polynomial (that is $\frac{P(x)}{Q(x)}$). Here are some examples:

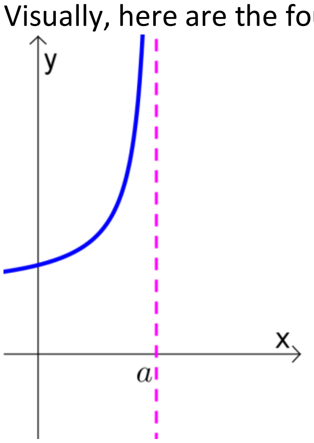


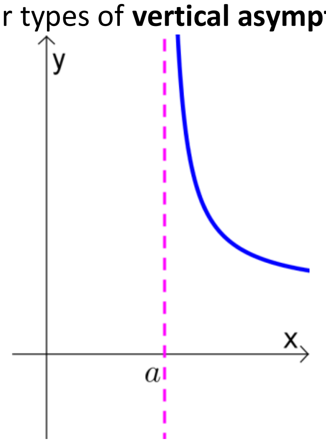




Most of the time, rational functions have vertical asymptotes and a horizontal asymptote.
An **asymptote** is basically a line/curve to which the graph of the function gets closer and closer.

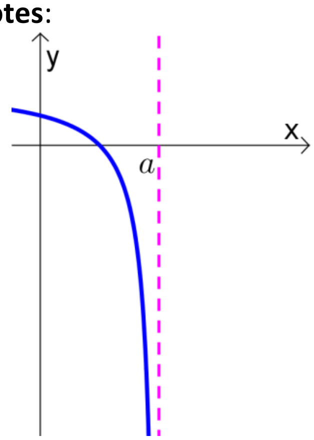
Visually, here are the four types of **vertical asymptotes**:



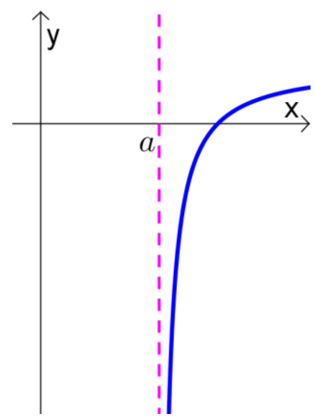

When $x \to a^-, y \to \infty$




When $x \to a^+, y \to \infty$




When $x \to a^-, y \to -\infty$



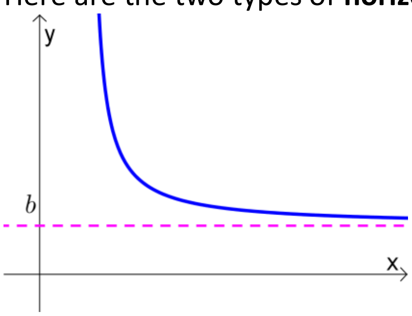

When $x \to a^+, y \to -\infty$

**Note:**
$x \to a^-$ means "$x$ approaches $a$ from the left"
$x \to a^+$ means "$x$ approaches $a$ from the right"

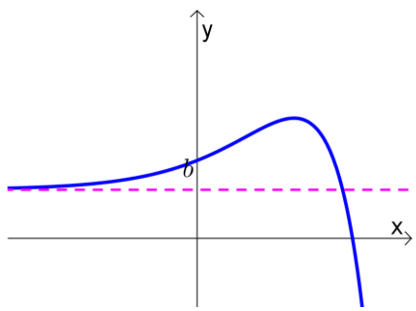
For rational functions, vertical asymptotes happen where the denominator is zero.
**Ex 1.**
Find the vertical asymptotes of the following functions. For each vertical asymptote, determine happens to $y$ as $x$ approaches from the left and from the right (that is, does $y \to \infty$ or $y \to -\infty$?).
$f(x) = \frac{x-1}{x+2}$

$g(x) = \frac{3x^2-2x-1}{2x^2+3x-2}$

Here are the two types of **horizontal asymptotes**:




When $x \to \infty$, $y \to b$




When $x \to -\infty$, $y \to b$

What is the horizontal asymptote (if any) of $\frac{P(x)}{Q(x)}$?

If deg(P) < deg (Q), then horizontal asymptote $y = 0$.

If deg(P) = deg(Q), then horizontal asymptote $y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$.

If deg(P) > deg(Q), then no horizontal asymptotes.

**Ex 2.**

Find the horizontal asymptotes (if any) of the following functions.

$f(x) = \frac{8x+5}{5x^2-3x+1}$

$g(x) = \frac{3x^2-2x-1}{2x^2+3x-2}$

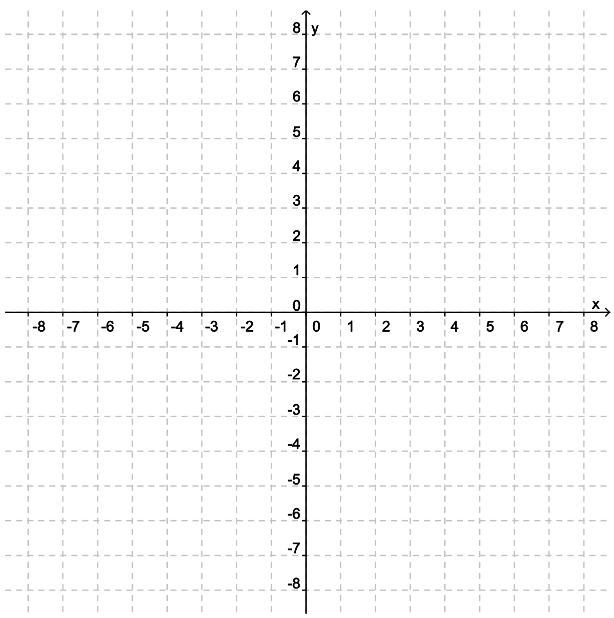
$h(x) = \frac{6x^3-x^2+2}{11x^2+4x-7}$

**Graphing Rational Functions**

1. **Factor** numerator and denominator.
2. Find $x$**-intercepts** and $y$**-intercept**.
3. Find **vertical asymptotes** (where denominator is 0).
4. Find **horizontal or slant asymptotes** (if any).
5. Graph (plot additional points as needed).

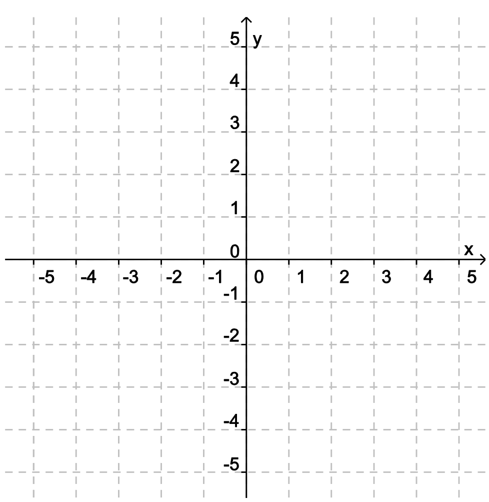
**Ex 3.**

Graph $f(x) = \frac{2x^2+7x-4}{x^2+x-2}$




**Ex 4.**

Graph $g(x) = \frac{x-3}{x^2+4x+4}$



Sometimes rational functions have **slant asymptotes** (also called **oblique asymptotes**), which are neither vertical nor horizontal.

**Ex 5.**

Graph $h(x) = \frac{x^2-4x-5}{x-3}$