Zeros of Polynomials

(covers parts of Sullivan 4.5 and 4.6)

Rational Zeros Theorem
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every rational zero of P is of the form $\frac{p}{q}$ where p is a factor of a_0 and q is a factor of a_n.

Ex 1.
Use the Rational Zeros Theorem to list all possible rational zeros of
$P(x) = 4x^5 - 5x^3 - 6x^2 - 2x + 3$.

Ex 2.
Factor and find all zeros of $P(x) = 2x^3 + x^2 - 13x + 6$.

Ex 3.

Factor and find all zeros of $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

Complex Numbers

Notice that $x^2 + 1 = 0$ does not have any real number solutions.
The **imaginary number** $\boldsymbol{i} = \sqrt{-\mathbf{1}}$ was introduced to create a solution (note that $i^2 = -1$).
With i, you can create a number system (called the **complex number system**) that is larger than the real number system. Here's how it's done:

Every complex number can be written in the form $\boldsymbol{a} + \boldsymbol{bi}$, where a is the **real part** and b is the **imaginary part**. Both a and b are real numbers. Note that any real number can be written as a complex number (just let $b = 0$).

Here are some examples of complex numbers: $\frac{-1+3i}{2}$ $\quad 5 \quad$ $-2i$

$(3 + 5i) + (4 - 2i) = \mathbf{7} + \mathbf{3}\boldsymbol{i}$
$(3 + 5i) - (4 - 2i) = -\mathbf{1} + \mathbf{7}\boldsymbol{i}$
$(3 + 5i)(4 - 2i) = 12 - 6i + 20i - 10i^2 = 12 + 14i + 10 = \mathbf{22} + \mathbf{14}\boldsymbol{i}$

The **conjugate** of $a + bi$ is ___________. To divide complex #'s, we can use the conjugate to help.

$$\frac{3+5i}{1-2i} = \frac{(3+5i)\cdot(1+2i)}{(1-2i)\cdot(1+2i)} = \frac{3+6i+5i+10i^2}{1+2i-2i-4i^2} = \frac{3+11i-10}{1+4} = \frac{-7+11i}{5} = -\frac{\mathbf{7}}{\mathbf{5}} + \frac{\mathbf{11}}{\mathbf{5}}\boldsymbol{i}$$

Fundamental Theorem of Algebra
Every polynomial with complex coefficients has at least one complex zero.

Complete Factorization Theorem
Every polynomial can be broken down into linear factors like this:
$P(x) = a(x - c_1)(x - c_2) \dots (x - c_n)$ (here, $a, c_1, c_2, \dots, c_n$ are complex numbers)

Ex 4.
Find the complete factorization and all zeros of $P(x) = x^3 - 3x^2 + x - 3$.

Ex 5.

Find the complete factorization and all zeros of $P(x) = x^3 - 2x + 4$.

Zeros Theorem

Every degree n polynomial has exactly n zeros (here, a zero of multiplicity k is counted k times).

Ex 6.

Find the complete factorization and all five zeros of $P(x) = 3x^5 + 24x^3 + 48x$.

Conjugate Zeros Theorem
Complex zeros come in conjugate pairs. That is, if $a + bi$ is a zero, then $a - bi$ is also a zero.
(Note: This is only true for polynomials with real coefficients, which is mostly what you see anyway.)

Ex 7.
Find a degree 3 polynomial with zeros 2 and $1 + 3i$.

Note: Since polynomials are continuous (can be drawn without picking up your pencil), if you find two function values, $P(a)$ and $P(b)$, that have opposite signs, then $P(x)$ must cross the x-axis at some x-value c between a and b. This is called the **Intermediate Value Theorem (for Polynomials)**. The same thing is true for all continuous functions.