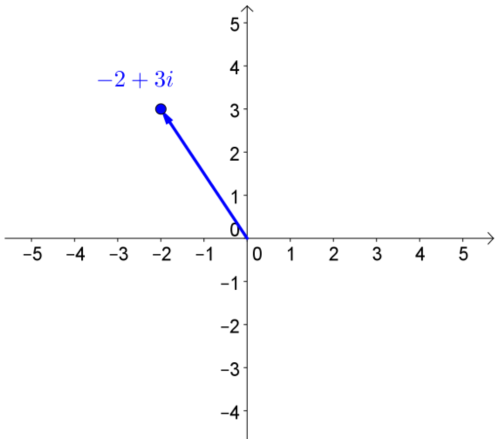
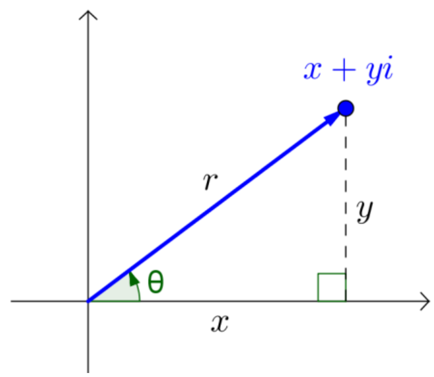
Polar (Trig) Form of Complex Numbers

You can graph complex numbers $x + yi$ on a two-dimensional plane, just like you graph points (x, y). Below is an example of how to graph $-2 + 3i$ on the complex plane:



Because of this, the form $x + yi$ is called **rectangular form**. Any point in the complex plane can be represented in rectangular form.

But we can also represent any point $x + yi$ in the complex plane in **polar form** (also called trigonometric form), using our familiar radius r and an angle θ. See below picture for the relationship between rectangular and polar form:



Using trig and the Pythagorean Theorem, we get:

$\boldsymbol{x = r\cos\theta}$, $\boldsymbol{y = r\sin\theta}$

$\boldsymbol{r = \sqrt{x^2 + y^2}}$, $\boldsymbol{\tan\theta = \frac{y}{x}}$

So, $x + yi = r\cos\theta + (r\sin\theta)i = \boldsymbol{r(\cos\theta + i\sin\theta)}$. This is polar form.

Note: r is called the **absolute value** (or **modulus**), and θ is called the **argument**.

Note: The book writes $r \operatorname{cis}\theta$ as shorthand for $r(\cos\theta + i\sin\theta)$, and you can too, but we're not going to in the notes.

Ex 1.

Express $2(\cos 300° + i\sin 300°)$ in rectangular form.

Ex 2.

Write $-\sqrt{3} + i$ in trigonometric form. Use radians for the angles.

Product Theorem
To multiply complex numbers in trig/polar form, you multiply the r's and add the θ's.
$[r_1(\cos\theta_1 + i\sin\theta_1)]\cdot[r_2(\cos\theta_2 + i\sin\theta_2)] = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Ex 3.
Find the product of $3(\cos 45^\circ + i\sin 45^\circ)$ and $2(\cos 135^\circ + i\sin 135^\circ)$. Write the result in rectangular form.

Quotient Theorem
To divide two complex numbers in trig/polar form, you divide the r's and subtract the θ's.
$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

Ex 4.
Find the quotient of $10(\cos(-60^\circ) + i\sin(-60^\circ))$ and $2(\cos 165^\circ + i\sin 165^\circ)$. Write the result in rectangular form.

De Moivre's Theorem
$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

Ex 5.
Find $\left(1 + i\sqrt{3}\right)^8$ and express the result in rectangular form.

nth Root Theorem

The nth roots of $r(\cos\theta + i\sin\theta)$ are given by $\sqrt[n]{r}(\cos\alpha + i\sin\alpha)$, where $\alpha = \frac{\theta}{n} + \frac{360°}{n}\cdot k$ or $\alpha = \frac{\theta}{n} + \frac{2\pi}{n}\cdot k$ for $k = 0,1,2,\ldots,n-1$. Note that there are n nth roots.

Ex 6.
Find the two square roots of $4i$. Write the roots in rectangular form.

Ex 7.
Find all fourth roots of $-8 + 8i\sqrt{3}$. Write the roots in rectangular form.

Ex 8.
Find all complex solutions of $x^5 - 1 = 0$. Leave your solutions in trig form with degrees. Graph your solutions as vectors in the complex plane.

Practice

1. Write $-3i$ in trigonometric form. Use degrees for the angles.

2. Find the product and quotient of $24(\cos 150^\circ + i \sin 150^\circ)$ and $2(\cos 30^\circ + i \sin 30^\circ)$. Write both results in rectangular form.

3. Find all cube roots of 27. Leave answers in trig form with radians.

4. Find all complex solutions of $x^6 + i = 0$. Leave your solutions in trig form with degrees. Graph your solutions as vectors in the complex plane.

Q: A bus driver was heading down a street in Walnut. He went right past a stop sign without stopping, went the wrong way on a one-way street, and then went on the left side of the road past a cop car. The cop did nothing, because he didn't break any traffic laws. Why not?