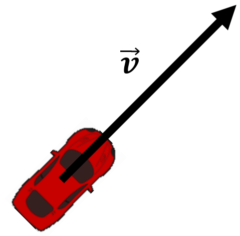
# Vectors

Suppose a car is heading NE (northeast) at 60 mph.
We can use a vector to help draw a picture (see right).

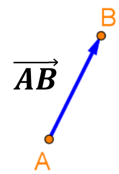



A **vector** consists of two parts:
1. a magnitude (ex: 60 mph)
2. a direction (ex: NE)

By contrast, a **scalar** only has a magnitude (ex: 60 mph).

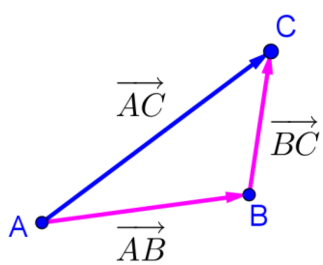
Examples of vectors include: displacement, velocity, acceleration, and force.
Examples of scalars include: distance, speed, time, and volume.

A vector has an **initial point** and a **terminal point**.
The magnitude is the length of the vector.

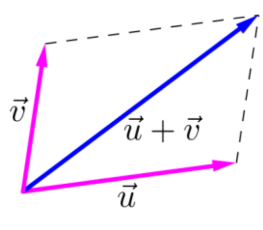



Two vectors are **equal** if they have the same magnitude and direction.
So, all of the vectors to the right are equal.

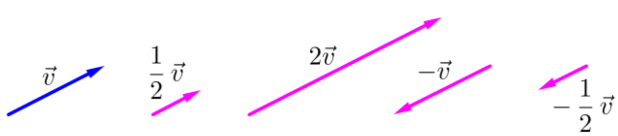
To add two vectors visually, put them tip to tail and make a triangle.




Or put them tail to tail and make a parallelogram.




You can stretch and shrink vectors by multiplying by a scalar. Negative values make the vector go in the opposite direction.



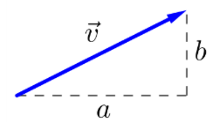
**Ex 1.**
Two forces of 15 and 22 newtons act on a point in the plane. If the angle between the forces is $100°$, find the magnitude of the resultant force.

The **equilibrant** of a resultant vector is the negative of that vector. So, the equilibrant of the resultant $\vec{u}$ is $-\vec{u}$.

**Ex 2.**
Two rescue vessels are pulling a broken-down motorboat toward a boathouse with forces of 840 lb and 960 lb. The angle between these forces is $24.5°$. Find the direction and magnitude of the equilibrant.

**Ex 3.**
A 3000-lb car is parked on a driveway that is inclined $15°$ to the horizontal. Find the magnitude of the force required to keep the car from rolling down the driveway.

## Vectors in a Coordinate System




In a two-dimensional coordinate system, we can represent vectors using two components: an $x$-component (horizontal component) and a $y$-component (vertical component). Here's the notation: $\vec{v} = \langle a, b \rangle$

The **magnitude** (or **length**) of a vector $\vec{v} = \langle a, b \rangle$ is $|\vec{\boldsymbol{v}}| = \sqrt{\boldsymbol{a}^2 + \boldsymbol{b}^2}$.
The **direction** angle $\theta$ satisfies $\tan\theta = \frac{b}{a}$.

**Ex 4.**
Find the magnitude and direction angle of $\vec{u} = \langle -\sqrt{3}, 1 \rangle$.

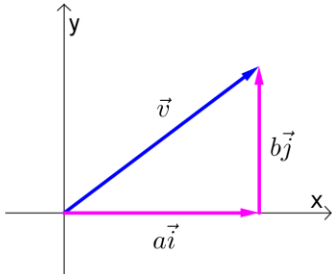
To add/subtract vectors, you just add/subtract their components.
ex: $\langle 3, -1 \rangle + \langle -4, 7 \rangle = \langle 3 + (-4), -1 + 7 \rangle = \langle -1, 6 \rangle$

To multiply a vector by a scalar, you just "distribute" the scalar to each component.
ex: $-3\langle 2, 5 \rangle = \langle -6, -15 \rangle$

**Ex 5.**
If $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -1, 2 \rangle$, find $2\vec{u} - 3\vec{v}$.

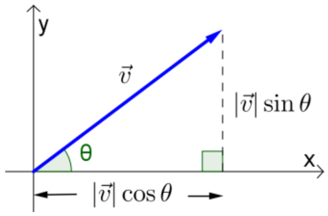
**Notes:**
$\vec{0} = \langle 0, 0 \rangle$ is called the **zero vector**.
Vectors with length 1 are called ___________________.
ex: $\vec{w} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ is a unit vector since $|\vec{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

Two special unit vectors are given their own letters: $\vec{\boldsymbol{\imath}} = \langle \mathbf{1}, \mathbf{0} \rangle$ and $\vec{\boldsymbol{\jmath}} = \langle \mathbf{0}, \mathbf{1} \rangle$
We can represent any vector in terms of $\vec{\imath}$ and $\vec{\jmath}$: $\vec{v} = \langle a, b \rangle = a\vec{\imath} + b\vec{\jmath}$




**Ex 6.**
Write $\vec{u} = \langle 5, -8 \rangle$ in terms of $\vec{\imath}$ and $\vec{\jmath}$.

It is often useful to break a vector into its horizontal and vertical components.




**Ex 7.**
A vector $\vec{v}$ has a direction angle of 60° and a magnitude of 8. Find the magnitudes of the horizontal and vertical components, and write $\vec{v}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$.

## Dot Product

The **dot product** of $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$ is: $\boldsymbol{\vec{u} \cdot \vec{v} = ac + bd}$

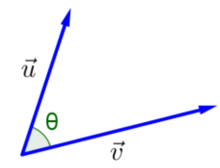
**Ex 8.**
If $\vec{u} = \langle 4, -2 \rangle$ and $\vec{v} = \langle 1,3 \rangle$, find $\vec{u} \cdot \vec{v}$.

**Properties of Dot Products**
1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$
4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

Here are proofs of the first and fourth properties (here, let $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$):
1. $\vec{u} \cdot \vec{v} = ac + bd = ca + db = \vec{v} \cdot \vec{u}$
4. $|\vec{u}|^2 = \left(\sqrt{a^2 + b^2}\right)^2 = a^2 + b^2 = \vec{u} \cdot \vec{u}$

**The Dot Product Theorem**

$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| \cos \boldsymbol{\theta}$ (here, $\theta$ is the angle between $\vec{u}$ and $\vec{v}$)




The Dot Product Theorem can help us find the angle between two vectors.

**Ex 9.**

Find the angle between $\vec{u} = \langle 4, -2 \rangle$ and $\vec{v} = \langle 1,3 \rangle$.

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are ____________________ if and only if ____________.

Why? If $\vec{u}$ and $\vec{v}$ are perpendicular, then $\theta = \frac{\pi}{2}$ so $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \frac{\pi}{2} = 0$.

And if $\vec{u} \cdot \vec{v} = 0$, then $|\vec{u}||\vec{v}| \cos \theta = 0$, so $\theta$ must be $\frac{\pi}{2}$ and so $\vec{u}$ and $\vec{v}$ are perpendicular.

Another word for perpendicular is ________________.

**Ex 10.**

Determine whether $\vec{u} = \langle 3,5 \rangle$ and $\vec{v} = \langle 2, -8 \rangle$ are orthogonal.

**Ex 11.**

Determine whether $\vec{u} = \langle 2,1 \rangle$ and $\vec{v} = \langle -1,2 \rangle$ are orthogonal.

## Practice

1. A 200-lb box is on a ramp. If a force of 80 lbs is just sufficient to keep the box from sliding, find the angle of inclination of the plane. (Assume no friction on the ramp.)

2. Suppose $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 0,3\rangle$.
   a) Find $|\vec{u}|$.
   b) Find $5\vec{v} - 2\vec{u}$.
   c) Write $\vec{v}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$.

3. Find the magnitudes of the horizontal and vertical components of $\vec{v}$ given that $|\vec{v}| = 50$ and $\theta = 120°$.

4. Find the direction (in degrees) of $\vec{v} = \langle -2, -3\rangle$.

5. Let $\vec{u} = \langle 3, -1\rangle$ and $\vec{v} = \langle 2,4\rangle$.
   a) Find $\vec{u} \cdot \vec{v}$.
   b) Find the angle between $\vec{u}$ and $\vec{v}$.

6. Determine if $\vec{u} = \langle 2, -5\rangle$ and $\vec{v} = \langle 10, -4\rangle$ are orthogonal.

Q: A man while looking at a photograph said, "Brothers and sisters have I none. That man's father is my father's son." Who was the person in the photograph?