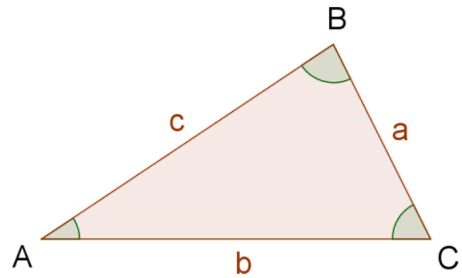
The Law of Cosines

When we have Case 3 (SAS) or Case 4 (SSS), we can use the Law of Cosines to solve triangles.

The Law of Cosines



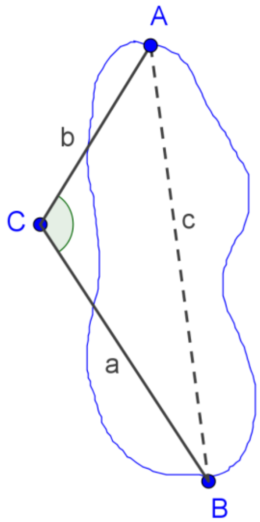
$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$

Depending on how you label the sides, you can also write it:
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Ex 1.
Solve triangle ABC if $A = 42°$, $b = 6.5$ m, and $c = 15$ m.

Ex 2.

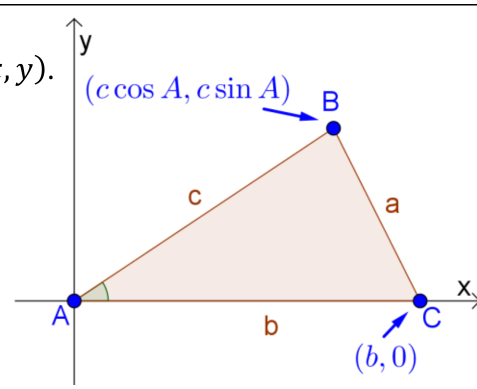
A surveyor wishes to find the distance between two inaccessible points A and B on opposite sides of a lake. While standing at point C, she finds that $b = 259$ m, $a = 423$ m, and angle ACB measures $132°40'$. Find the distance from A to B.



Let's prove the Law of Cosines …

Proof that $a^2 = b^2 + c^2 - 2bc\cos A$:

In the triangle to the right, we can represent the point B with as (x, y).
Then $\cos A = \frac{x}{c}$ and $\sin A = \frac{y}{c}$. So, $x = c\cos A$ and $y = c\sin A$.
Thus, point B is $(c\cos A, c\sin A)$.
And point C is $(b, 0)$.



Using the distance formula between B and C, we get:
$a = \sqrt{(c\cos A - b)^2 + (c\sin A - 0)^2}$
Square both sides: $a^2 = (c\cos A - b)^2 + (c\sin A)^2$
Multiply out: $a^2 = c^2\cos^2 A - 2bc\cos A + b^2 + c^2\sin^2 A$
Rearrange terms and factor out a c^2: $a^2 = b^2 + c^2(\sin^2 A + \cos^2 A) - 2bc\cos A$
Since $\sin^2 A + \cos^2 A = 1$, we arrive at our claim:
$a^2 = b^2 + c^2 - 2bc\cos A$
QED

FYI, there's a formula to get the area of a triangle given the lengths of its sides.

Heron's Area Formula

$\boldsymbol{Area} = \sqrt{\boldsymbol{s(s-a)(s-b)(s-c)}}$ (where $s = \frac{1}{2}(a+b+c)$)

Practice

1. Solve triangle ABC if $a = 25$ ft, $b = 43$ ft, and $c = 59$ ft. (Hint: you can use the Law of Cosines to get one angle, then the Law of Sines to get another angle.)

Q: What word can you make by adding letters to each side of XYG? (Hint: add one letter to the left side, and two letters to the right side.)