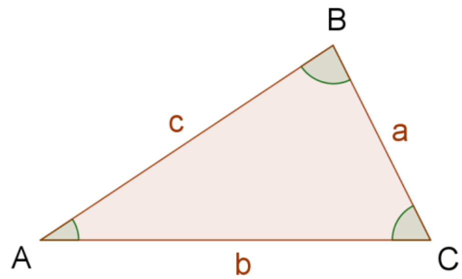
The Law of Sines

Q: We know how to solve right triangles using trig, but how can we use trig to solve <u>any</u> triangle?
A: The Law of Sines (7.1/7.2) and The Law of Cosines (7.3).

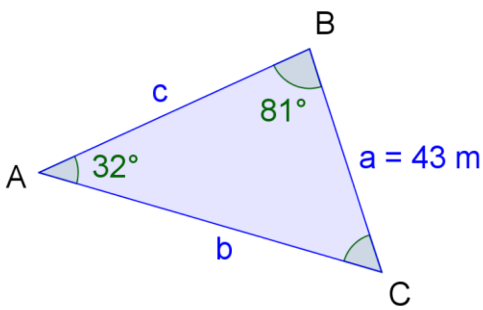
The Law of Sines



$$\frac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin \boldsymbol{C}}$$

(also written $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$)

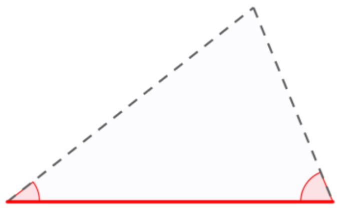
Ex 1.
Solve triangle ABC if $A = 32°$, $B = 81°$, and $a = 43$ m.

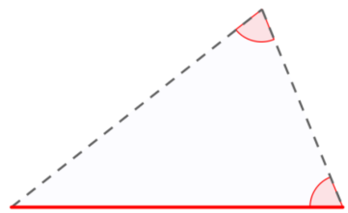


Ex 2.
The bearing of a lighthouse from a ship was found to be $N\ 52°\ W$. After the ship sailed 5.8 km due south, the new bearing was $N\ 23°\ W$. Find the distance between the ship and the lighthouse at each location.

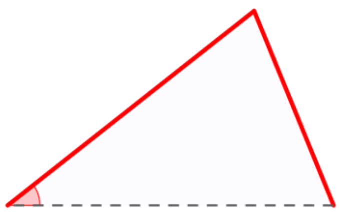
To solve a triangle, we've needed to know some of its sides and angles. We can classify the possibilities like this (A = Angle, S = Side):

Case 1: ASA or SAA

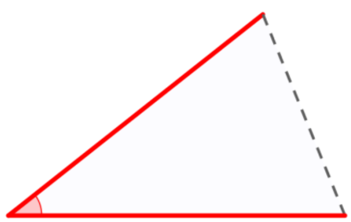




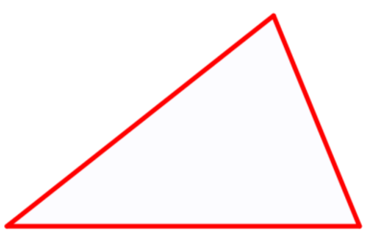
Case 2: SSA



Case 3: SAS



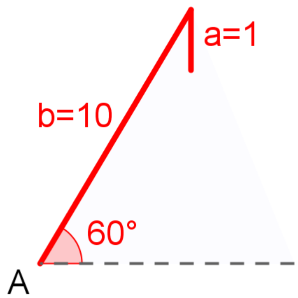
Case 4: SSS



Note: The Law of Sines helps us solve cases 1 and 2. The Law of Cosines helps us solve cases 3 and 4.

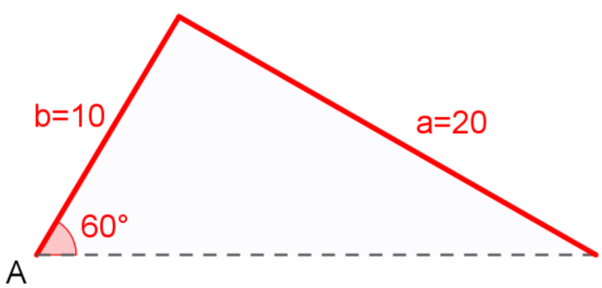
Pain in the SSA

What if we're given the following info and asked to solve the triangle?
$a = 1, b = 10, A = 60^\circ$
This is Case 2 (SSA). But if we try to draw the triangle we get…



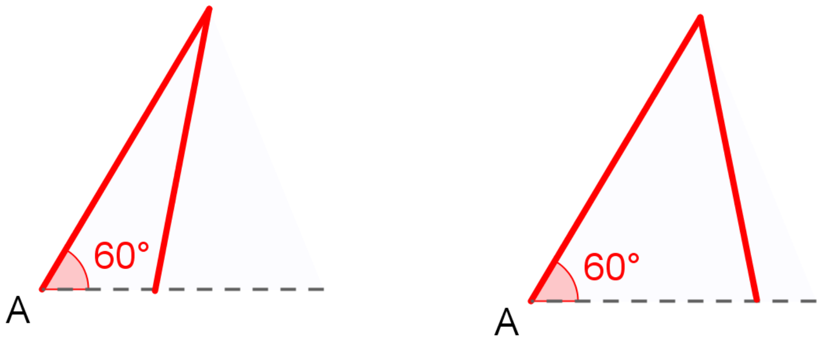
Side a is too short! We can see this with the Law of Sines.

If we make side a longer, then we can make a triangle:
$a = 20, b = 10, A = 60^\circ$



Now using the Law of Sines we get…

You can imagine if side a is in between (and just right), we might get two possible triangles:



This is called the **ambiguous case**, and it only happens with Case 2 (SSA).

Ex 3.
Solve the triangle ABC if it exists.
$b = 25.0, c = 30.0, B = 25.0°$

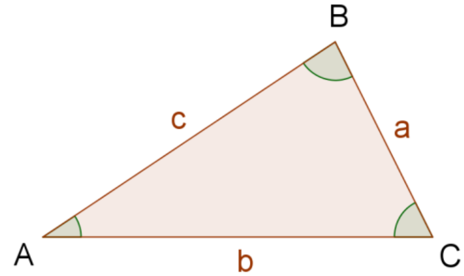
Area of a Triangle (SAS)

$\boldsymbol{Area = \frac{1}{2}bc\sin A}$

Note that b and c are the two sides that make the angle A.

Also note that the area formula could be written:

$Area = \frac{1}{2}ab\sin C$ or $Area = \frac{1}{2}ac\sin B$



Proof that $\boldsymbol{Area = \frac{1}{2}bc\sin A}$:

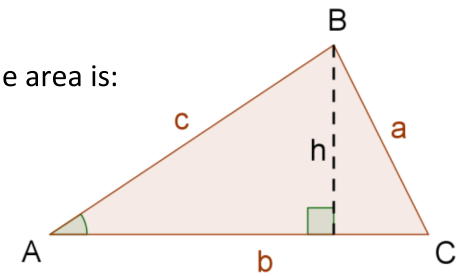
We can drop a perpendicular down to the base b (see right). So, the area is:

$Area = \frac{1}{2}bh$

But $\sin A = \frac{h}{c}$ so that $h = c\sin A$. Replacing h with $c\sin A$, we get:

$Area = \frac{1}{2}bc\sin A$

QED



Let's go back and prove the Law of Sines as well…

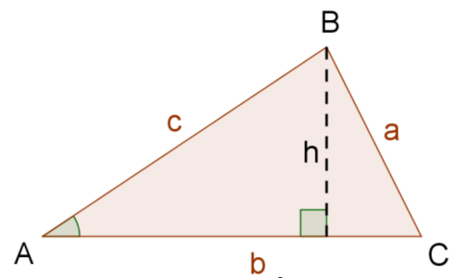
Proof that $\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$:

Using the triangle to the right, we get: $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$

So, $h = c\sin A$ and $h = a\sin C$

Thus, $c\sin A = a\sin C$

Dividing, we get, $\frac{c}{\sin C} = \frac{a}{\sin A}$



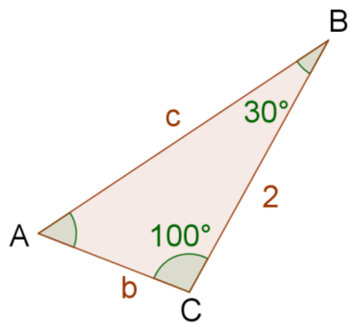
By making perpendicular segments from A and C, and using similar reasoning, we get $\frac{b}{\sin B} = \frac{c}{\sin C}$ and $\frac{a}{\sin A} = \frac{b}{\sin B}$.

Thus, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

QED

Practice

1. Solve triangle ABC if $B = 30°$, $C = 100°$, and $a = 2$ ft.



2. Solve triangle ABC if $A = 61.4°$, $a = 35.5$ cm, and $b = 39.2$ cm.

3. Find the area of the triangle ABC.
   $B = 55°10', a = 34.0\text{ ft}, c = 42.0\text{ ft}$

Q: When can you add two to eleven and get one as the correct answer?