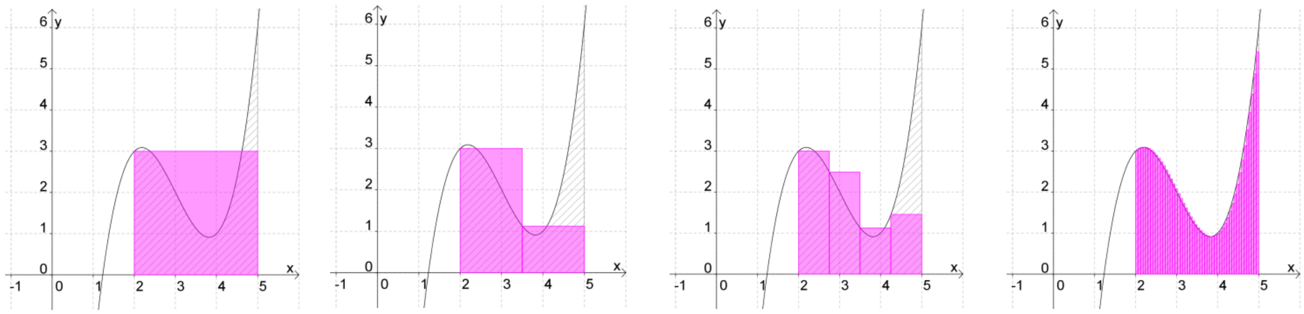
The Definite Integral and the Fundamental Theorem of Calculus

Q: How can we find the area under any curve?

For example, let's try to find the area under $f(x) = x^3 - 9x^2 + 25x - 19$ from $x = 2$ to $x = 5$. We can approximate the area with rectangles.



1 rectangle 2 rectangles 4 rectangles 50 rectangles

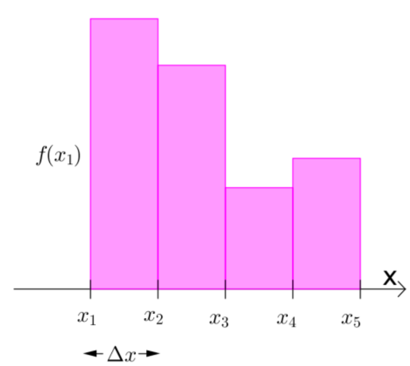
The more rectangles we use, the closer our approximation is to the actual area under the curve.

Above, we used the height of the function to determine the height of each rectangle.

So, if the left sides of our rectangles are at $x_1, x_2, \ldots, x_n$ and each rectangle had width Δx, then the areas of the rectangles are $f(x_1)\Delta x, f(x_2)\Delta x, \ldots, f(x_n)\Delta x$.

So, the sum of the areas of the rectangles (called a **Riemann sum**) is:

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$= [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$



As the number of rectangles (n) increases, the above sum gets closer to the true area. That is,

$$\text{Area under curve} = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

Notation:

$$\int_a^b f(x)dx = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x$$

The above integral is called a **definite integral**, and a and b are called the **lower and upper limits of integration**.

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on the interval $a \leq x \leq b$, then

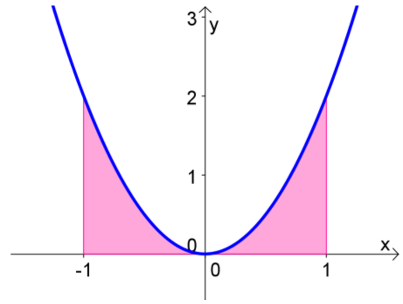
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \leq x \leq b$.

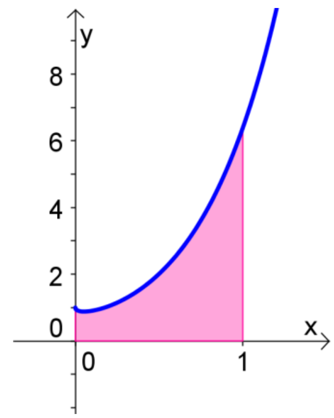
Ex 1.

Evaluate the given definite integrals using the fundamental theorem of calculus.

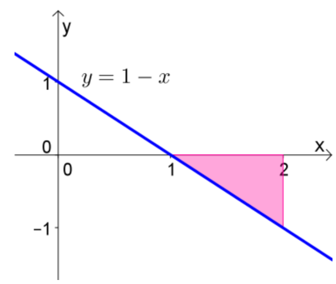
$$\int_{-1}^{1} 2x^2\,dx$$



$$\int_{0}^{1} (e^{2x} - \sqrt{x})\,dx$$



Note: Definite integrals can evaluate to *negative* numbers. For example, $\int_1^2 (1-x)\,dx = -\frac{1}{2}$



Just like with indefinite integrals, terms can be integrated separately, and constants can be pulled out. There are also a few more properties listed below.

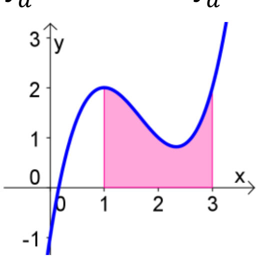
$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

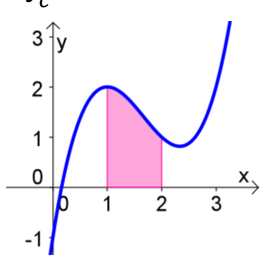
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

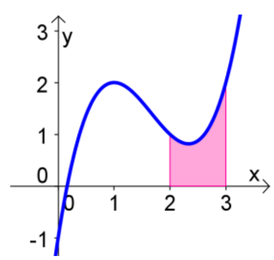
$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \text{(here, } c \text{ is any real number)}$$







Ex 2.

Let $f(x)$ and $g(x)$ be functions that are continuous on the interval $-3 \le x \le 5$ and that satisfy

$$\int_{-3}^{5} f(x)\,dx = 2 \qquad \int_{-3}^{5} g(x)\,dx = 7 \qquad \int_{1}^{5} f(x)\,dx = -8$$

Use this information along with the rules for definite integrals to evaluate the following.

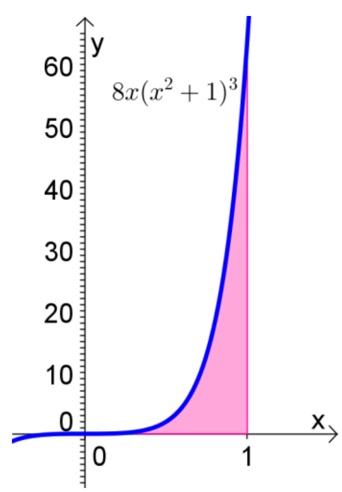
$$\int_{-3}^{5} [2f(x) - 3g(x)]\,dx$$

$$\int_{-3}^{1} f(x)\,dx$$

Ex 3.

Evaluate:

$$\int_{0}^{1} 8x(x^2+1)^3\,dx$$



Net Change

If we have $Q'(x)$, then we can calculate the **net change** in $Q(x)$ as x goes from a to b with an integral:

$$Q(b) - Q(a) = \int_a^b Q'(x)\,dx$$

Ex 4.

Suppose marginal cost is $C'(x) = 3(x-4)^2$ dollars per unit when the level of production is x units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units?

Practice

1. Evaluate:

a) $\int_1^2 \left(x^2 + \frac{1}{x}\right) dx$

b) $\int_0^1 \frac{3x^2}{(x^3+1)^2}\, dx$

Q: What is harder to catch the faster you run?