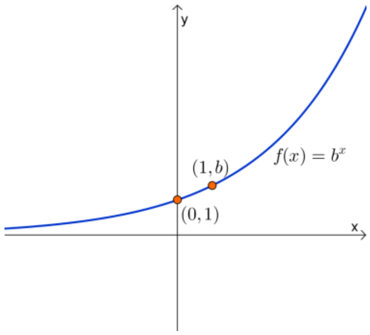
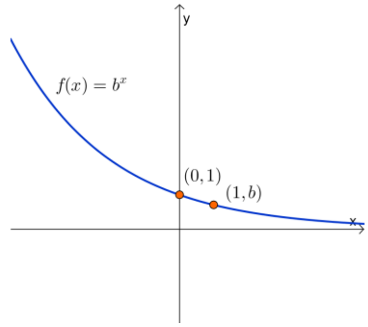
Exponential and Logarithmic Functions

Some things just don't grow linearly, they grow exponentially (ex: population, compound interest). To model such behavior, we use the **exponential function**, $f(x) = b^x$. b is the base ($b > 0$, $b \neq 1$).



$\boldsymbol{b > 1}$
ex: $f(x) = 2^x$



$\boldsymbol{0 < b < 1}$
ex: $f(x) = \left(\frac{1}{2}\right)^x$

The most important base in calculus is e (called the natural exponential base), which is defined as:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.718281828459045 \ldots$$

This can be interpreted as the amount of money you'd have in an account if you invested \$1 at 100% interest rate per year for one year, where interest is compounded continuously.
(In general, the continuously compounded interest formula is $A = Pe^{rt}$, and the regular compound interest formula is $A = P\left(1 + \frac{r}{n}\right)^{nt}$.)

Properties of exponents:

$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$ $\quad b^{-n} = \frac{1}{b^n}$ $\quad b^0 = 1$ $\quad b^x = b^y$ if and only if $x = y$

$b^x b^y = b^{x+y}$ $\quad \frac{b^x}{b^y} = b^{x-y}$ $\quad (b^x)^y = b^{xy}$ $\quad (ab)^x = a^x b^x$ $\quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Ex 1.
Evaluate:

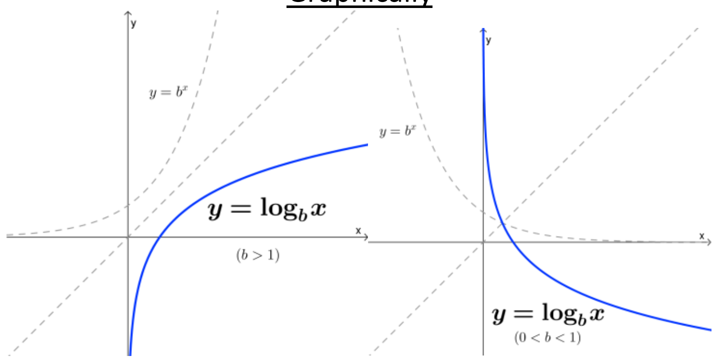
$81^{3/4}$

$\left(e^3 e^{1/2}\right)^{2/3}$

The **logarithmic function** $f(x) = \log_b x$ is the ______________ of the exponential function.
What does that mean?

Definition
$\log_b x = y \iff b^y = x$
(for $x > 0$)

Graphically



Algebraically
$\log_b(b^x) = x$
$b^{\log_b x} = x$

$\log x$ means $\log_{10} x$ $\ln x$ means $\log_e x$

Ex 2.
Evaluate.

$\log_2 16 =$ $\log 1000 =$

$\ln e =$ $\ln \sqrt{e} =$

Properties of logarithms:

$\log_b u = \log_b v$ iff $u = v$ $\log_b(uv) = \log_b u + \log_b v$ $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$

$\log_b u^r = r \log_b u$ $\log_b 1 = 0$ $\log_b b = 1$

Ex 3.
Expand: $\log_2\left(\frac{x^2y^3}{z^5w^8}\right)$

Ex 4.

Expand: $\ln\left(\frac{x^3\sqrt{1-x}}{y^2}\right)$

Solving Exponential and Logarithmic Equations

Ex 5.

Solve: $e^{-x} - 2xe^{-x} = 0$

Ex 6.

Solve: $6 = 4 + 10e^{-4x}$

Ex 7.

Solve: $5\ln(x+7) = 15$

To evaluate logs with any base, you can change them to natural logs with this formula: $\log_b a = \frac{\ln a}{\ln b}$

ex: $\log_5 2 = \frac{\ln 2}{\ln 5} \approx 0.4307$

Practice

1. Evaluate:

a) $125^{2/3}$

b) $\left(\frac{e^2}{\sqrt{e}}\right)^{4/3}$

c) $\ln e^{5x}$

2. Expand: $\ln\left(\frac{x^3}{(x-2)\sqrt{1+x^2}}\right)$

3. Solve: $5 + 5e^{-0.1x} = 15$

4. Solve: $2\ln(3x) - 1 = 7$

Q: A man rode his horse into town on Tuesday. Two days later he rode home on Tuesday. How is this possible?