Math 130

7.3 – Geometric Sequences and Series

Geometric Sequences

A **geometric sequence** is one where you *multiply* a fixed # to get the next term.

ex: $1, 2, 4, 8, 16, \ldots$ is a geometric sequence. The **common ratio** r is ____.

In general, for geometric sequences, $\boldsymbol{a_n = a_1 r^{n-1}}$

Ex 1.

Find a_5 and a_n for the following geometric sequence.

$a_1 = -2, \quad r = 3$

Ex 2.

Find a_5 and a_n for the following geometric sequence.

$6400, 1600, 400, 100, \ldots$

Ex 3.

Find a_1 and r for the following geometric sequence.

$a_3 = 5, \quad a_8 = \frac{1}{625}$

Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence.

Here's a geometric series: $2 + 6 + 18 + 54 + 162 + 486$

And here's how to find the sum of the first n terms of a geometric series:

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}$ (*)

Multiply both sides by r to get:

$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n$ (**)

Subtract (**) from (*):

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ -(rS_n &= \phantom{a_1 +{}} a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + a_1 r^n) \end{aligned}$$

And we get:

$S_n - rS_n = a_1 - a_1 r^n$

Solve for S_n:

$S_n(1 - r) = a_1(1 - r^n)$

$$\boldsymbol{S_n = \frac{a_1(1-r^n)}{1-r}} \quad \textbf{(where } \boldsymbol{r \neq 1}\textbf{)}$$

Ex 4.

Use the formula for S_n to find the sum of the first five terms of the following geometric sequence:

$3, 12, 48, \ldots$

Ex 5.

Find:

$$\sum_{i=1}^{8} 4 \cdot 5^i$$

Infinite Geometric Series

Consider the infinite geometric sequence:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

Here is the related series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Does this series approach any particular number as we add more and more terms?

Let's see what happens to the "partial sums":

$$S_1 = \mathbf{1}$$

$$S_2 = 1 + \frac{1}{2} = \mathbf{\frac{3}{2}}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \mathbf{\frac{7}{4}}$$

$$S_4 = \cdots = \mathbf{\frac{15}{8}}$$

$$S_5 = \cdots = \mathbf{\frac{31}{16}}$$

$$S_6 = \cdots = \mathbf{\frac{63}{32}}$$

$S_n =$_______________ (this is a formula for each partial sum)

As n becomes larger and larger, S_n gets closer and closer to ______.

We simply say that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots =$_______

Ex 6.

Evaluate $4+\frac{4}{5}+\frac{4}{25}+\cdots$

For a geometric series in general, we know

$S_n=\frac{a_1(1-r^n)}{1-r}$

Note that, if $-1<r<1$, then r^n approaches 0 as n goes to infinity. So, as n goes to infinity, S_n approaches $\frac{a_1(1-0)}{1-r}=\frac{\boldsymbol{a_1}}{\boldsymbol{1-r}}$.

Ex 7.

Find the sum.

$$\sum_{i=1}^{\infty}\left(\frac{2}{5}\right)\left(-\frac{1}{3}\right)^{i-1}$$

$$\sum_{k=1}^{\infty}(0.9)^k$$

Q: A man leaves home and, after making three left turns, he ends up back at home, and finds two masked men waiting for him. What is happening?