Math 130

4.2 – Exponential Functions

Recall:

$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

$4^{1.5} =$

It turns out that exponents can be any real # (rational or **irrational**).

ex: $2^{\sqrt{3}} = 3.321997\ldots$

We can get this by taking closer and closer approximations to $\sqrt{3}$:

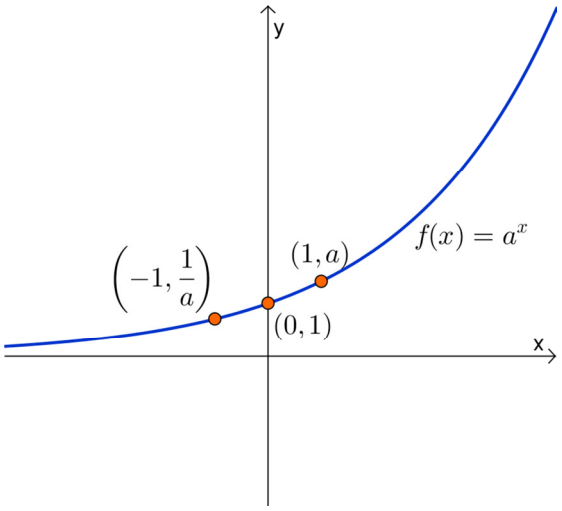
$2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17} = 3.249010\ldots$

$2^{1.73} = 2^{173/100} = \left(\sqrt[100]{2}\right)^{173} = 3.317278\ldots$

$2^{1.732} = 2^{1732/1000} = \left(\sqrt[1000]{2}\right)^{1732} = 3.321880\ldots$

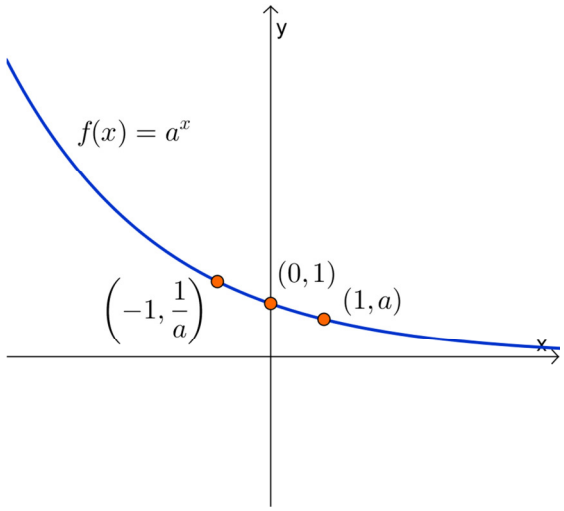
This allows us to define a function $f(x) = 2^x$ that has a domain of all real #'s.

In general, the **exponential function with base a** is $\boldsymbol{f(x) = a^x}$ (where $a > 0$ and $a \neq 1$).



$\boldsymbol{a > 1}$

ex: $f(x) = 2^x$



$\boldsymbol{0 < a < 1}$

ex: $f(x) = \left(\frac{1}{2}\right)^x$

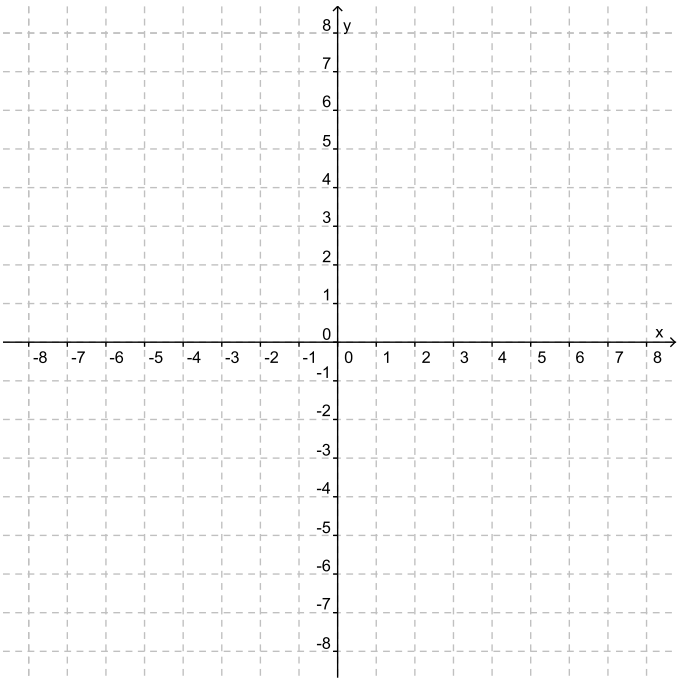
What is the horizontal asymptote?

Any vertical asymptotes?

What is the range?

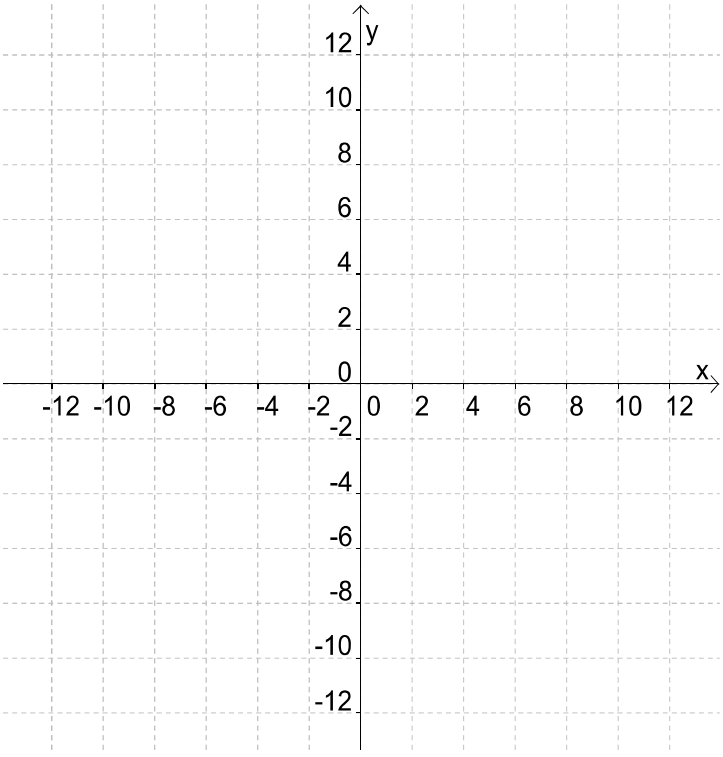
Ex 1.

Graph $f(x) = -2^x$.



Ex 2.

Graph $f(x) = 3^{x-2} - 4$.



Exponential Equations

Ex 3.

Solve $\left(\frac{1}{5}\right)^{x} = 125$

Ex 4.

Solve $3^{x+1} = 9^{x-3}$

Ex 5.

Solve $b^{5/2} = 243$

Compound Interest

Suppose you invest P dollars at 5% interest (that compounds annually).

Year	Calculations	Expression for \$ in account
0		P
1	$P + 0.05P$	$P(1 + 0.05)$
2	$P(1 + 0.05) + 0.05 \cdot P(1 + 0.05) = P(1 + 0.05)(1 + 0.05)$	$P(1 + 0.05)^2$
3	$P(1 + 0.05)^2 + 0.05 \cdot P(1 + 0.05)^2 = P(1 + 0.05)^2(1 + 0.05)$	$P(1 + 0.05)^3$
4	$P(1 + 0.05)^3 + 0.05 \cdot P(1 + 0.05)^3 = P(1 + 0.05)^3(1 + 0.05)$	$P(1 + 0.05)^4$
…	…	…
t		$P(1 + 0.05)^t$

So, when compounding annually with initial principal P and interest rate r, here's a formula for the amount of money in the account t years later:

$A = P(1 + r)^t$

If interest is compounded n times per year, then the formula becomes:

$A = P\left(1 + \frac{r}{n}\right)^{tn}$

(Note: A is sometimes called **future value**, and P the **present value**.)

Ex 6.

Suppose $\$2500$ is deposited in an account paying 6% per year compounded semiannually (twice per year). Find the amount in the account after 10 years with no withdrawals. Also, how much interest is earned over the 10-year period?

Ex 7.

How much would you need to deposit today at 8% compounded quarterly to have $\$10000$ in your account in 20 years?

Ex 8.

Suppose you only have $\$1000$ to invest right now. What annual interest rate is necessary for the money to increase to $\$10000$ in 20 years?

Continuous Compounding, and *e*

The more often interest is compounded, the more money you make. But it has a limit…

Suppose $\$1$ is invested at 100% interest per year for one year. Then the interest formula is:

$$A = P\left(1 + \frac{r}{n}\right)^{tn} = \left(1 + \frac{1}{n}\right)^{n}$$

What happens when we compound more and more frequently?

n (# of times we compound)	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
1,000,000	2.71828

$\left(1+\frac{1}{n}\right)^n$ approaches a fixed number, 2.718281828459045…, which we call $\boldsymbol{e}$.

As a result, we can talk about calculating how money grows if we **compound continuously**.
The interest formula becomes:
$\boldsymbol{A = Pe^{rt}}$

Ex 9.
Suppose $\$8000$ is deposited in an account paying 5% interest compounded continuously for 6 years. Find the total amount in the account at the end of 6 years.

Q: What is it that you will break even when you name it?